Response Time Properties of Some Asynchronous Circuits

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Abstract

New results including distinct forward and reverse delays.

Here is the original theorem for reference.
Results in this paper.

- Worst-case response times for pipelines with more general cells than previously described. Each cell may have a forward and reverse latency, and each cell may have a different set of delays from other cells.
- Best-case and worst-case amortized response times for pipelines.
- The above results given in terms of cycle time rather than response time.

0.1 Comparison with previous results

1 Communication Behaviours

Consider the cell in Figure 1 from a linear array. At interface i, the communication behaviour for

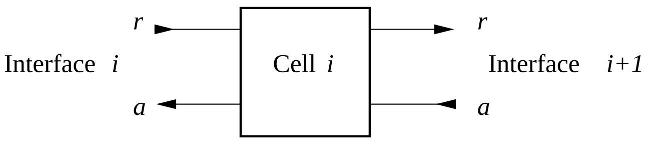

Figure 1: One cell

the handshakes can be given by $*[r_i?;a_i!]$. That is, we take an input request, $r_i?$, and then we output

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an acknowledgement $a_i!$. (? denotes input, while ! denotes output.) This behaviour then repeats (indicated by $*[..]$). At interface $i+1$ the communication behaviour is given by $*[r_{i+1}!; a_{i+1}?]$.

Cell i synchronizes the handshakes at interface i and $i+1$. In general, the communication behaviour for a cell can take many different forms. We restrict ourselves to behaviours for cells where for every n handshakes on interface i there is one handshake on interface $i+1$. More precisely, the communication behaviours we look at are given by

$$(r_i?;\ *[\,(a_i!;\ r_i?)^n \| (r_{i+1}!;\ a_{i+1}?)\,])$$

In other words, every occurrence of the segment $r_{i+1}!;\ a_{i+1}?$ takes place in parallel with the n handshakes $a_i!;\ r_i?$. Note that this differs from the behaviour given in a previous paper which was

$$(r_i?;\ *[\,(a_i!;\ r_i?)^{n-1};\ ((a_i!;\ r_i?) \| (r_{i+1}!;\ a_{i+1}?))\,])$$

This change is made for a number of reasons. When $n = 1$, both behaviours are identical, so the only change occurs when $n > 1$. The new behaviour is easier to understand, and requires less explanation than the old behaviour. If one derived an upperbound for response times given the old behaviour, this bound would also be an upperbound for the new behaviour. In this paper, we are also looking for lowerbounds on response times, so the previous advantage disappears.

The deciding factor in choosing the new behaviour is its considerably greater mathmatical simplicity. Not only do the results resolve into simpler formulae, with simpler conditions, but some subtle difficulties which were glossed over in previous work have now ceased to be a problem. These subtleties relate to the assymmetric nature of the old behaviour, which particularly manifested themselves when $n > 2$. When $n > 2$ there are large strings of events at the first cell (i.e., at level 0) with no interaction from cells deeper in the pipeline. These effects reduce the bounds and were ignored in previous work.

For $n = 1$ we have a communication behaviour that can be represented by

$$(r_i?;\ *[\,(a_i!;\ r_i?) \| (r_{i+1}!;\ a_{i+1}?)\,])$$

An unfolded process graph for $n = 1$ for a cell is given in Figure 2.

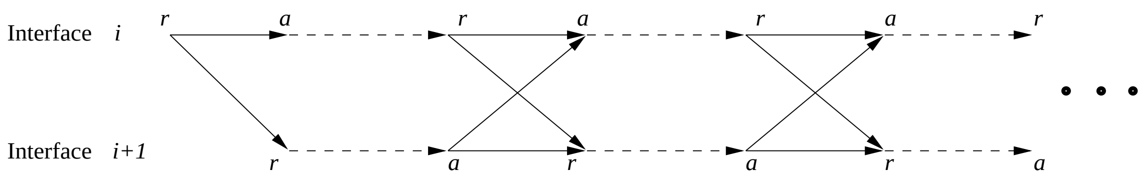

Figure 2: Unfolded communication behaviour for a cell

Let us denote the k-th occurrence of event a_i by (a_i, k), where $k \geq 0$. A process graph gives a precedence relation for occurrences of events. A precedence between occurrences of events e_0 and e_1 is denoted by $e_0 \to e_1$. For example, in Figure 2 we have the precedences

$$(r_i, j) \to (a_i, j) \quad \text{and} \quad (a_{i+1}, j-1) \to (a_i, j)$$

These precedences indicate that occurrence j of event a_i can only happen after occurrence j of event r_i and occurrence $j-1$ of event a_{i+1} has happened.

Solid arrows indicate precedence relations brought about by the cell. Dashed arrows indicate precedence relations brought about by the environment of the cell.

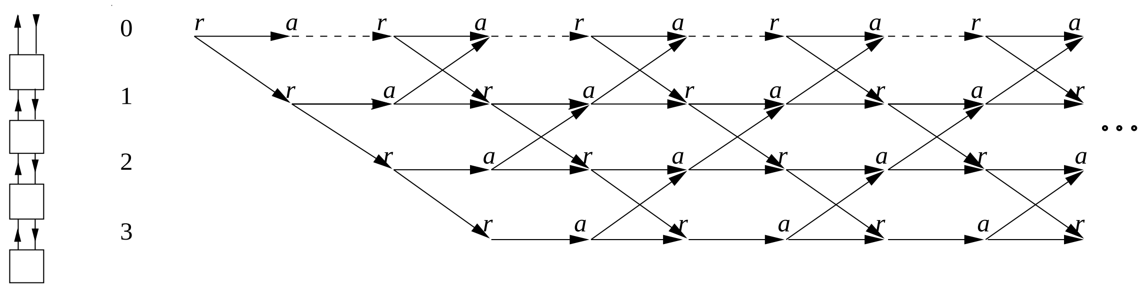

Figure 3: Unfolded communication behaviour for a linear array of four cells and $n = 1$

If we consider the complete network of 3 non-end cells, one end cell, and the environment of the pipeline, then we can depict its behaviour in a process graph (Figure 3), where all edges represent direct precedences brought about by some component in the network. The dashed arrows represent direct precedences caused by the environment of the pipeline.

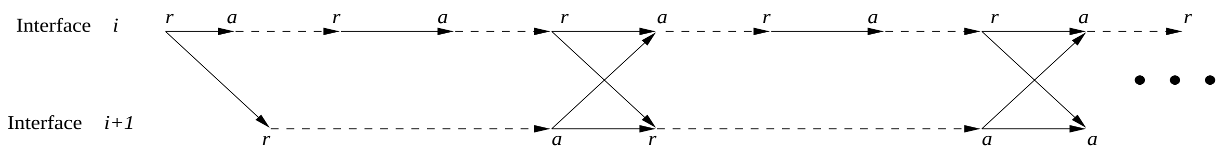

Figure 4: Unfolded communication behaviour for a cell with $n = 2$

For $n = 2$ we have a communication behaviour that can be represented by

$$(r_i?;\ *[\,(a_i!;\ r_i?)^2 \| (r_{i+1}!;\ a_{i+1}?)\,])$$

An unfolded process graph for $n = 2$ for a cell is given in Figure 4 and the corresponding graph for a linear array of four cells is given in Figure 5.

2 The Delay Model

Let c be a node in a (unfolded) process graph with c an output of cell h. Our delay model prescribes that any timing assignment T of the process graph must satisfy the following constraints.

$$T_{pred}(c) + \delta_{h,f} \leq T(c) \leq T_{pred}(c) + \Delta_{h,f}$$

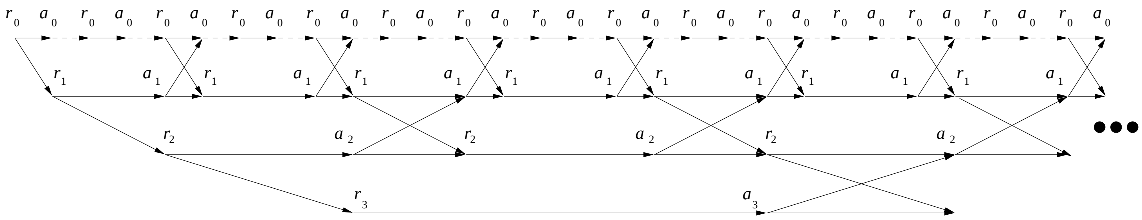

Figure 5: Part of an unfolded process graph for a linear array of four cells with $n = 2$.

if node c is a request, and

$$T_{pred}(c) + \delta_{h,r} \leq T(c) \leq T_{pred}(c) + \Delta_{h,r}$$

if node c is an acknowledgment.

$T(c)$ is the time assigned to node c and

$$T_{pred}(c) = \max\{T(b) \mid b \text{ is a direct predecessor of c}\}$$

This condition stipulates that the delay is always measured from the time $T_{pred}(c)$, when the *last* directly preceding event occurs, to the actual occurrence of the event c. The constraints prescribe that "forward" delays through cell h have lower bound $\delta_{h,f}$ and upper bound $\Delta_{h,f}$, that is

$$\delta_{h,f} \leq \text{cell } h \text{ forward delays} \leq \Delta_{h,f}$$

Similarly, "reverse" delays for cell h are bounded by $\delta_{h,r}$ and $\Delta_{h,r}$.

We must be clear about which delays are associated with cell h. Cell 0 is controls the "forward" edges $r_0 \to r_1$ and $a_1 \to r_1$. Likewise, cell 0 controls the "reverse" edges $r_0 \to a_0$ and $a_1 \to a_0$. Therefore cell h controls edges

$$\left.\begin{array}{l} r_h \to r_{h+1} \\ a_{h+1} \to r_{h+1} \end{array}\right\} \text{forward latencies.}$$

$$\left.\begin{array}{l} r_h \to a_h \\ a_{h+1} \to a_h \end{array}\right\} \text{reverse latencies.}$$

The end-cell is special in that it has only reverse latencies associated with it, since the only edge through the end cell is the "reverse" edge $r_L \to a_L$. Similarly, the environment delays are only forward latencies as the environment edges are of the form $a_0 \to r_0$. Essentially they can be thought of as $\delta_{-1,f}$ and $\Delta_{-1,f}$. We label the lowerbound and upperbound on the environment delays δ_e and Δ_e respectively.

3 A Timing Property

Before we formulate some properties about response times, we define the concept of *response depth* of an occurrence of an event. Let the following path exist in a process graph for a behaviour B

$$(a_i, j_i) \to (a_{i-1}, j_{i-1}) \to ...(a_{k+1}, j_{k+1}) \to (a_k, j_k)$$

with $i \geq k$ and occurrences $j_i < j_{i-1} < j_{i-2} < ... < j_{k+1} < j_k$. That is, the acknowledgment a_k depends on a_i via a sequence of acknowledgments, one for each interface. Let furthermore T be a timing assignment for behaviour B. We say that (a_k, j_k) has *response depth* $i - k$ for behaviour B and timing assignment T iff the path $(a_i, j_i) \rightarrow (a_{i-1}, j_{i-1}) \rightarrow ...(a_{k+1}, j_{k+1}) \rightarrow (a_k, j_k)$ is a critical path, and the edge $(a_{i+1}, j_{i+1}) \rightarrow (a_i, j_i)$ does not exist, or, if it does exist, it is not critical. We call this critical path the "response path" for acknowledgment (a_k, j_k). For example, if we have a response path with $k = 0$, and $i = L$, then we have a critical path of acknowledgments from the end-cell to the environment, with response depth L.

We now state the main theorem.

Theorem 1 *If the response depth of (a_i, j_i) is $k, 1 \leq k \leq L - i$, then*

$$T(a_i, j_i) - T(a_i, j_i - n^k) \leq \Delta_{i+k,r} + \Delta_{i+k-1,f} + \sum_{h=0}^{k-1}(\Delta_{i+h,r} - \delta_{i+h,r})$$

□

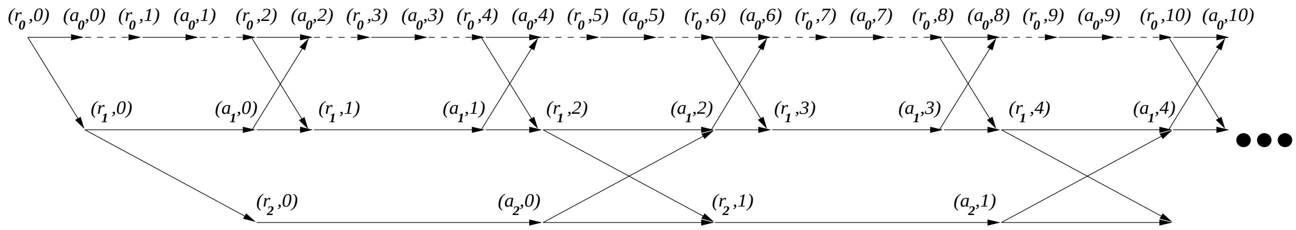

Figure 6: Part of an unfolded process graph for a linear array of three cells (i.e. $L = 2$) with $n = 2$. The vertices are labelled with occurrence indices.

This theorem says that if acknowledgment (a_i, j_i) has a response depth of $k, 1 \leq k \leq L - i$, then the duration of the n^k handshake cycles at interface i that end in (a_i, j_i) is at most $\Delta_{i+k,r} + \Delta_{i+k-1,f} + \sum_{h=i}^{k-1}(\Delta_{i+h,r} - \delta_{i+h,r})$. The factor n^k in the above equation will prove useful, because we give a linear bound to the duration between two events that have an exponential number of events between them (if $n > 1$).

Here is a brief example of the main theorem. Suppose $n = 2$, $k = 2$, $L = 2$, and that we are looking at event a_0 $(i = 0)$. Refer to Fig 6. We will choose $j_0 = 10$. Therefore we wish to show that

$$\begin{aligned}
T(a_0, 10) - T(a_0, 10 - 2^2) &\leq \Delta_{i+k,r} + \Delta_{i+k-1,f} + \sum_{h=0}^{k-1}(\Delta_{i+h,r} - \delta_{i+h,r}) \\
T(a_0, 10) - T(a_0, 10 - 4) &\leq \Delta_{0+2,r} + \Delta_{0+2-1,f} + \sum_{h=0}^{2-1}(\Delta_{0+h,r} - \delta_{0+h,r}) \\
T(a_0, 10) - T(a_0, 6) &\leq \Delta_{2,r} + \Delta_{1,f} + \sum_{h=0}^{1}(\Delta_{h,r} - \delta_{h,r})
\end{aligned}$$

$$T(a_0,10) - T(a_0,6) \quad \leq \quad \Delta_{2,r} + \Delta_{1,f} + (\Delta_{1,r} - \delta_{1,r}) + (\Delta_{0,r} - \delta_{0,r})$$

Since $k = 2$, we know that $(a_0, 10)$ is dependent on an acknowledgment 2 levels deeper in the pipeline, viz., $(a_2, 1)$. So, tracing further back, $(a_0, 10)$ must be dependent on $(r_2, 1)$. In turn $(r_2, 1)$ is dependent on both $(r_1, 2)$ and $(a_2, 0)$. Suppose for the moment that the critical edge is the $(a_2, 0) \to (r_2, 1)$.

The path from $(a_2, 0)$ to $(a_0, 10)$ consists of the following four edges: $(a_2, 0) \to (r_2, 1)$, $(r_2, 1) \to (a_2, 1)$, $(a_2, 1) \to (a_1, 4)$, and $(a_1, 4) \to (a_0, 10)$. Maximimizing this path means assigning the following delays respectively for each edge: $\Delta_{1,f}$,$\Delta_{2,r}$,$\Delta_{1,r}$, and $\Delta_{0,r}$. The path from $(a_2, 0)$ to $(a_0, 6)$ consists of the following two edges: $(a_2, 0) \to (a_1, 2)$, and $(a_1, 2) \to (a_0, 6)$. Minimimizing this path means assigning the following delays respectively for each edge: $\delta_{1,r}$, and $\delta_{0,r}$. So we get

$$T(a_0,10) - T(a_0,10-2^2) \quad \leq \quad (T(a_2,0) + \Delta_{1,f} + \Delta_{2,r} + \Delta_{1,r} + \Delta_{0,r}) - (T(a_2,0) + \delta_{1,r} + \delta_{0,r})$$

which is equivalent to

$$T(a_0,10) - T(a_0,6) \quad \leq \quad \Delta_{2,r} + \Delta_{1,f} + (\Delta_{1,r} - \delta_{1,r}) + (\Delta_{0,r} - \delta_{0,r})$$

If the critical edge is $(r_1, 2) \to (r_2, 1)$ the same result can be derived.

Before we prove the main theorem, we give some lemmas.

Lemma 2 *Let b be an input to cell h, i.e., either a request r_h or an acknowledgment a_{h+1}. Let c be an output of cell h, i.e. either a request r_{h+1} or an acknowledgment a_h.*

(i) *If we have the direct dependency $b \to c$, where c is a request then*

$$T(c) - T(b) \geq \delta_{h,f} \quad \textit{and, hence } T(b) - T(c) \leq -\delta_{h,f}$$

(ii) *If we have the direct dependency $b \to c$, where c is an acknowledgment then*

$$T(c) - T(b) \geq \delta_{h,r} \quad \textit{and, hence } T(b) - T(c) \leq -\delta_{h,r}$$

Lemma 3 *For any $k, 0 \leq k \leq L$, if the response depth of (a_k, j_k) is 0, then*

$$T(a_k, j_k) - T(r_k, j_k) \leq \Delta_{k,r}$$

Lemma 4 *If the dependency $(a_{i+1}, j_{i+1}) \to (a_i, j_i)$ exists, and if $(a_{i+1}, j_{i+1} - n^{k-1})$ exists for some $k > 0$ (where for our purposes the event exists if the occurrence index $j_{i+1} - n^{k-1} \geq 0$). Then the dependency $(a_{i+1}, j_{i+1} - n^{k-1}) \to (a_i, j_i - n^k)$ also exists.*

□ *(Proofs omitted).*

Lemma 2 says that if we know of the existence of a dependency, then the separation between the source and target event must be at least δ. Lemma 3 says that if an acknowledgment to a request to cell k, depends only on cell k itself and not on cells $k+1, \ldots L$, then the only delay incurred is the delay of cell k. Lemma 4 says that the occurrence index changes by a factor of n between levels when we keep the dependency pattern the same. In the proof below, we assume that the occurrence indices are large enough to satisfy the second condition of Lemma 4.

With the preceding lemmas we can now prove the main theorem.

Proof of Theorem 1. By induction on k.
Basis. Assume that (a_i, j_i) has a response depth of $k = 1$. Let $(a_{i+1}, j_{i+1}) \rightarrow (a_i, j_i)$, then (a_{i+1}, j_{i+1}) has response depth 0. We have a dependency graph as in Figure 7. From the definition

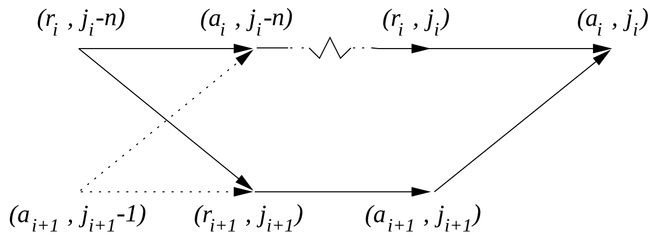

Figure 7: Example of paths used in the proof. The jagged line consists of $2n-1$ edges.

of $T_{pred}(c)$ and the dependency graph in Figure 7, we have $T_{pred}(r_{i+1}, j_{i+1}) = T_{pred}(a_i, j_i - 1)$. In the proof below we take $T_{pred} = T_{pred}(a_i, j_i - 1)$. We wish to derive an upper bound for $T(a_i, j_i) - T(a_i, j_i - n)$. We observe

$T(a_i, j_i) - T(a_i, j_i - n)$

$=$ $\{$ Break path into 4 edges. $\}$

$(T(a_i, j_i) - T(a_{i+1}, j_{i+1})) + (T(a_{i+1}, j_{i+1}) - T(r_{i+1}, j_{i+1})) + (T(r_{i+1}, j_{i+1}) - T_{pred}) + (T_{pred} - T(a_i, j_i - n))$

$\leq$ $\{$ (a_i, j_i) has positive response depth so timing dependency exists. Also, ack/ack pair is a "reverse" delay. $\}$

$\Delta_{i,r} + (T(a_{i+1}, j_{i+1}) - T(r_{i+1}, j_{i+1})) + (T(r_{i+1}, j_{i+1}) - T_{pred}) + (T_{pred} - T(a_i, j_i - n))$

$\leq$ $\{$ (a_{i+1}, j_{i+1}) has resp. depth 0, Lemma 3 $\}$

$\Delta_{i,r} + \Delta_{i+1,r} + (T(r_{i+1}, j_{i+1}) - T_{pred}) + (T_{pred} - T(a_i, j_i - n))$

$=$ $\{$ $T_{pred} = T_{pred}(a_i, j_i - 1) = T_{pred}(r_{i+1}, j_{i+1})$ $\}$

$\Delta_{i,r} + \Delta_{i+1,r} + (T(r_{i+1}, j_{i+1}) - T_{pred}(r_{i+1}, j_{i+1})) + (T_{pred} - T(a_i, j_i - n))$

$\leq$ $\{$ From definition of timing assignments and because req/req is a forward delay $\}$

$\Delta_{i,r} + \Delta_{i+1,r} + \Delta_{i,f} + (T_{pred} - T(a_i, j_i - n))$

$\leq$ { Lemma 2, part (ii) }

$\Delta_{i,r} + \Delta_{i+1,r} + \Delta_{i,f} + (-\delta_{i,r})$

$=$ { calc. }

$\Delta_{i+1,r} + \Delta_{i,f} + (\Delta_{i,r} - \delta_{i,r})$

$=$ { calc. }

$\Delta_{i+k,r} + \Delta_{i+k-1,f} + \sum_{h=0}^{k-1}(\Delta_{i+h,r} - \delta_{i+h,r})$

Step. Assume the theorem holds for $k > 0$ and that (a_i, j_i) has response depth $k+1$. Let $(a_{i+1}, j_{i+1}) \rightarrow (a_i, j_i)$, then (a_{i+1}, j_{i+1}) has response depth k. We observe

$T(a_i, j_i) - T(a_i, j_i - n^{k+1})$

$=$ { Break path into three: edge, path, edge. }

$((T(a_i, j_i)) - T(a_{i+1}, j_{i+1})) + (T(a_{i+1}, j_{i+1}) - T(a_{i+1}, j_{i+1} - n^k)) + (T(a_{i+1}, j_{i+1} - n^k) - T(a_i, j_i - n^{k+1}))$

$\leq$ { (a_{i+1}, j_{i+1}) has resp. depth k, induction hyp. }

$((T(a_i, j_i)) - T(a_{i+1}, j_{i+1})) + \Delta_{i+1+k,r} + \Delta_{i+1+k-1,f} + \sum_{h=0}^{k-1}(\Delta_{i+1+h,r} - \delta_{i+1+h,r}) + (T(a_{i+1}, j_{i+1} - n^k) - T(a_i, j_i - n^{k+1}))$

$\leq$ { (a_i, j_i) has positive response depth, and ack/ack pair is a reverse latency }

$\Delta_{i,r} + \Delta_{i+k+1,r} + \Delta_{i+k,f} + \sum_{h=1}^{k}(\Delta_{i+h,r} - \delta_{i+h,r}) + (T(a_{i+1}, j_{i+1} - n^k) - T(a_i, j_i - n^{k+1}))$

$\leq$ { By Lemma 4, $(a_{i+1}, j_{i+1} - n^k) \rightarrow (a_i, j_i - n^{k+1})$ exists and we can apply Lemma 2 (ii) }

$\Delta_{i,r} + \Delta_{i+k+1,r} + \Delta_{i+k,f} + \sum_{h=1}^{k}(\Delta_{i+h,r} - \delta_{i+h,r}) + (-\delta_{i,r})$

$\leq$ { calc. }

$\Delta_{i+k+1,r} + \Delta_{i+k,f} + \sum_{h=0}^{k}(\Delta_{i+h,r} - \delta_{i+h,r})$

□

4 Worst-Case Response Time

We have the following theorem.

Theorem 5 *The worst-case response time RT for any pipeline (as defined in Section 3 operating under a delay model of Section 4) is bounded from above as*

$$RT \;\leq\; \max(\{\Delta_{k,r} + \Delta_{k-1,f} + \sum_{h=0}^{k-1}(\Delta_{h,r} - \delta_{h,r}) - (\delta_{0,r} + \delta_e)n^k + \delta_{0,r} \mid 1 \leq k \leq L\} \cup \{\Delta_{0,r}\})$$

Proof. We have to find an upper bound for $T(a_0, j) - T(r_0, j)$ over all $j \geq 0$ and all valid delay distributions. Each acknowledgment has a response depth k, where $0 \leq k \leq L$. For a response depth $k = 0$, we have from Lemma 3, $T(a_0, j) - T(r_0, j) \leq \Delta_{0,r}$. For a response depth of $k > 0$, we use the result of the previous Section. We observe

$$
\begin{aligned}
& T(a_0, j) - T(r_0, j) \\
= \quad & \{ \text{ calc. } \} \\
& (T(a_0, j) - T(a_0, j - n^k)) + (T(a_0, j - n^k) - (T(r_0, j)) \\
\leq \quad & \{ \ (a_0, j) \text{ has response depth } k, \text{ Theorem 1 } \} \\
& \Delta_{i+k,r} + \Delta_{i+k-1,f} + \textstyle\sum_{h=0}^{k-1}(\Delta_{i+h,r} - \delta_{i+h,r}) + (T(a_0, j - n^k) - (T(r_0, j)) \\
\leq \quad & \{ \ i = 0 \ \} \\
& \Delta_{k,r} + \Delta_{k-1,f} + \textstyle\sum_{h=0}^{k-1}(\Delta_{h,r} - \delta_{h,r}) + (T(a_0, j - n^k) - (T(r_0, j))
\end{aligned}
$$

Notice that there are n^k handshakes at level 0 between occurrences (a_0, j) and $(a_0, j - n^k)$. The handshakes at level 0 experience delays through the environment and through cell 0. Assuming that the environment's minimum delay is δ_e (for edge $a_0 \to r_0$) and cell zero's minimum delay (for a reverse latency edge $r_0 \to a_0$)is $\delta_{0,r}$, the minimum duration of one cycle through environment and cell is $\delta_{0,r} + \delta_e$. Therefore the minimum duration from $T(a_0, j - n^k)$ to $T(r_0, j)$ is $(\delta_{0,r} + \delta_e)n^k - \delta_{0,r}$. This then leads to the inequality

$$
T(a_0, j) - T(r_0, j) \leq \Delta_{k,r} + \Delta_{k-1,f} + \sum_{h=0}^{k-1}(\Delta_{h,r} - \delta_{h,r}) - (\delta_{0,r} + \delta_e)n^k + \delta_{0,r}
$$

Maximizing these upper bounds over all $k, 0 \leq k \leq L$ gives the desired result.
□

Note also that this bound may not necessarily be tight. For example, the proof of theorem 1 relies on lemma 2 to show that an edge must have at least delay $\delta_{h,r}$. What we don't know is if this edge is a critical edge. The edge in question is an $a_{h+1} \to a_h$ edge. a_h also has the predecessor r_h and therefore the edge $r_h \to a_h$ may give us a much larger separation between a_{h+1} and a_h than just $\delta_{h,r}$.

The bound above cannot be simplified further without further assumptions about the variables. Look at the $n = 1$ case.

$$
\begin{aligned}
RT \quad & \leq \quad \max(\{\Delta_{k,r} + \Delta_{k-1,f} + \sum_{h=0}^{k-1}(\Delta_{h,r} - \delta_{h,r}) - (\delta_{0,r} + \delta_e) \cdot 1 + \delta_{0,r} \mid 1 \leq k \leq L\} \cup \{\Delta_{0,r}\}) \\
& = \quad \max(\{\Delta_{k,r} + \Delta_{k-1,f} + \sum_{h=0}^{k-1}(\Delta_{h,r} - \delta_{h,r}) - \delta_e \mid 1 \leq k \leq L\} \cup \{\Delta_{0,r}\})
\end{aligned}
$$

I will define the following useful expressions.

$$
MC = max\{\delta_e + \Delta_{0,r}, \Delta_{0,f} + \Delta_{1,r}, \Delta_{1,f} + \Delta_{2,r}, \ldots, \Delta_{L-1,f} + \Delta_{L,r}\}
$$

Note that the environment delay is a lower bound, while all other values are upper bounds.

$$
mc = max\{\Delta_e + \delta_{0,r}, \delta_{0,f} + \delta_{1,r}, \delta_{1,f} + \delta_{2,r}, \ldots, \delta_{L-1,f} + \delta_{L,r}\}
$$

Note that the environment delay is an upper bound, while all other values are lower bounds.

Now, if $\Delta_{h,r} = \delta_{h,r}$ with $0 \leq h \leq L$ then

$$\begin{aligned}
RT &\leq \max(\{\Delta_{k,r} + \Delta_{k-1,f} + \sum_{h=0}^{k-1}(0) - \delta_e \mid 1 \leq k \leq L\} \cup \{\Delta_{0,r}\}) \\
&= \max(\{\Delta_{k,r} + \Delta_{k-1,f} - \delta_e \mid 1 \leq k \leq L\} \cup \{\Delta_{0,r}\}) \\
&= \max(\{\Delta_{k,r} + \Delta_{k-1,f} \mid 1 \leq k \leq L\} \cup \{\Delta_{0,r} + \delta_e\}) - \delta_e \\
&= MC - \delta_e
\end{aligned}$$

Now, we look at the $n > 1$ case. We assume that $\Delta_{h,r} \leq \Delta_{0,r} \cdot n^h$, $\Delta_{h-1,f} \leq \delta_e \cdot n^h$, and $\Delta_{h,r} = \delta_{h,r}$.

$$\begin{aligned}
RT &\leq \max(\{\Delta_{k,r} + \Delta_{k-1,f} + \sum_{h=0}^{k-1}(0) - (\delta_{0,r} + \delta_e)n^k + \delta_{0,r} \mid 1 \leq k \leq L\} \cup \{\Delta_{0,r}\}) \\
&= \max(\{\Delta_{0,r} \cdot n^k + \Delta_{k-1,f} - (\delta_{0,r} + \delta_e)n^k + \delta_{0,r} \mid 1 \leq k \leq L\} \cup \{\Delta_{0,r}\}) \\
&= \max(\{\Delta_{0,r} \cdot n^k + \Delta_{k-1,f} - (\Delta_{0,r} + \delta_e)n^k + \Delta_{0,r} \mid 1 \leq k \leq L\} \cup \{\Delta_{0,r}\}) \\
&= \max(\{\Delta_{k-1,f} - \delta_e \cdot n^k + \Delta_{0,r} \mid 1 \leq k \leq L\} \cup \{\Delta_{0,r}\}) \\
&\leq \max(\{0 + \Delta_{0,r} \mid 1 \leq k \leq L\} \cup \{\Delta_{0,r}\}) \\
&= \Delta_{0,r}
\end{aligned}$$

5 Amortized bounds for the $n = 1$ case.

We can also calculate bounds for the average response time. In a previous paper *** cite **** we described amortized response time as an upperbound on the average response time. That is, over many request/acknowledgment cycles, we calculate the worst possible delay caused by the pipeline, and divide this by the number of cycles. Here we also find a lower bound on the average response time by a similar technique. Over many request/acknowledgment cycles we calculate the *minimum* possible delay caused by the pipeline, and divide this by the number of cycles.

We have the following theorem.

Theorem 6 *The amortized response time AR for any pipeline (as defined in Section 3 operating under a delay model of Section 4), and with* $n = 1$*, is bounded as follows*

$$mc - \Delta_e \leq \quad AR \quad \leq MC - \delta_e$$

Proof. First we prove the upperbound.

From acknowledgment (a_0, j_0) there is a tight response path to some request (r_0, i_0) with $i_0 \leq j_0$ and on this tight response path there is no other event at level 0. Let this path be of length $h = 2k + 1$. Suppose there is either an edge $r_i \to r_{i+1}$ or an edge $a_{i+1} \to r_{i+1}$ in our path of length l. r_{i+1} is the output request signal of cell i. The path must pass back through cell i at some point which means there must be an acknowledgement a_{i+1} forthcoming from cell $i + 1$. So, for each edge on the path that ends in r_{i+1} there must be a corresponding edge that ends in a_{i+1}. For each pair of edges we have a maximum delay of $\Delta_{i,f} + \Delta_{i+1,r} \leq MC$. There are k requests in the path and so we have k pairs with total delay less than or equal to $k \cdot MC$. We also have one "loose" acknowledgment which pairs with the initial request by the environment. So the total delay on the path is bounded by $k \cdot MC + \Delta_{0,r}$.

We subtracting the minimum environment delay and then dividing by the number of cycles.

$$\begin{aligned} AR &\leq (k \cdot MC + \Delta_{0,r} - k\delta_e)/(k+1) \\ &= (\Delta_{0,r} + \delta_e - MC)/(k+1) + MC - \delta_e \\ &\leq ((MC) - MC)/(k+1) + MC - \delta_e \\ &= MC - \delta_e \end{aligned}$$

Next we prove the lower bound.

From acknowledgment (a_0, j_0) there are response paths to some request (r_0, i_0) with $i_0 \leq j_0$ and on these paths there is no other event at level 0. Select any such path P_i. Let this path be of length $h = 2k+1$. Given an edge corresponding to cell i, then the next edge must correspond to either cell $i-1$ or cell $i+1$. Therefore we can split the path into k pairs of edges plus one extra edge. We'll make the extra edge the $a_1 \rightarrow a_0$ edge that ends the path. Note that all P_i have k pairs of edges plus one extra $a_1 \rightarrow a_0$ edge.

We label every edge on path P_i with it's minimum value. Since the separation between (a_0, j_0) and (r_0, i_0) is dependent on the longest path between the two, i.e. the critical path between the two, then the separation depends on $max(\text{delay\_on\_path}(P_i) \mid \forall i)$. Note that $max(\text{delay\_on\_path}(P_i) \mid \forall i) \leq k \cdot mc + \delta_{0,r}$.

We do the usual trick of subtracting the environment delay and then dividing by the number of cycles. Since we are trying to find a lowerbound, we have to maximize the environment delay when we subtract it. That is, the environment delay cannot be more than $k\Delta_e$.

$$\begin{aligned} AR &\geq \max\{\text{delay\_on\_path}(P_i) - k\Delta_e)/(k+1) \mid \forall i, k\} \\ &= \max\{\text{delay\_on\_path}(P_i) + \Delta_e)/(k+1) \mid \forall i, k\} - \Delta_e \end{aligned}$$

Now examine the first term more closely.

$$\begin{aligned} \max\{\text{delay\_on\_path}(P_i) + \Delta_e)/(k+1) \mid \forall i, k\} &\leq \max\{k \cdot mc + \delta_{0,r} + \Delta_e)/(k+1) \mid \forall k\} \\ &\leq \max\{k \cdot mc + mc)/(k+1) \mid \forall k\} \\ &= \max\{(k+1) \cdot mc)/(k+1) \mid \forall k\} \\ &= mc \end{aligned}$$

This bound can be achieved when $k \rightarrow \infty$. When $k \rightarrow \infty$ then the length of the paths P_i also tend to infinity. On some path P_i there is an infinite number of cycles with delay equal to mc. This will clearly be the longest path. So when $k \rightarrow \infty$ then

$$\lim_{k \rightarrow \infty} \max(\text{delay\_on\_path}(P_i) \mid \forall i) = k \cdot mc + \delta_{0,r}$$

So from the above two equations, we can show that

$$\max\{\text{delay\_on\_path}(P_i) + \Delta_e)/(k+1) \mid \forall i, k\} = mc$$

Then we have

$$\begin{aligned} AR &\geq \max\{\text{delay\_on\_path}(P_i) + \Delta_e)/(k+1) \mid \forall i, k\} - \Delta_e \\ &= mc - \Delta_e \end{aligned}$$

□

6 AR for $n > 1$

Theorem 7 *Let the maximum delays for each cell be bounded as follows.*

$$\begin{aligned}\Delta_{i,f} &\leq \Delta_{0,f} \cdot n^i \\ \Delta_{i,r} &\leq \Delta_{0,r} \cdot n^i\end{aligned}$$

The amortized response time AR for any pipeline (as defined in Section 3 operating under a delay model of Section 4), and with $n > 1$, is bounded from above as follows

$$\text{(i)} \quad AR \leq \max\{\Delta_{0,r}, \Delta_{0,r} + \Delta_{0,f} - \delta_e\}$$

Instead, let the maximum delays be unbounded. Let the minimum delays for each cell be bounded as follows.

$$\begin{aligned}\delta_{i,f} &\leq \delta_{0,f} \cdot n^i \\ \delta_{i,r} &\leq \delta_{0,r} \cdot n^i\end{aligned}$$

The amortized response time AR for any pipeline (as defined in Section 3 operating under a delay model of Section 4), and with $n > 1$, is bounded from above as follows

$$\text{(ii)} \quad AR \geq \delta_{0,r}$$

Proof of (i). Let

$$N_k = (\sum_{i=0}^{k-1} n^i) = (n^k - 1)/(n - 1)$$

I define the following notation. A path $P \in [x, y]$ is a path in a dependency graph that consist of events between level x and level y. For example a path $P \in [0, L]$ would be an unrestricted set of paths. A path $P \in (x, y]$ is a path $P \in [x + 1, y]$, except for the end points of the path, which are events at level x. For example $P \in (0, 2]$ might be a path $r_0 \to r_1 \to r_2 \to a_2 \to a_1 \to a_0$. Finally, we call a path a *level x path*, if it consists of events only at level x. This is represented in the graphs by a horizontal set of edges. A path $P \in [x, x]$ would also describe a level x path,

Theorem 8 *Let $\Delta_{i,f} = \Delta_{0,f} \cdot n^i$ and $\Delta_{i,r} = \Delta_{0,r} \cdot n^i$ be given. Let paths P and P' exist between (r_h, i) and (a_h, j), given $h > 0$, and $j > i$. Let path $P \in (h, h + k]$ have maximal delays on each edge, and let P' be a level h path, also with maximal delays on each edge. Then the delay on P is the same as the delay on P'.*

In other words, if we have a path between request and acknowledgment events at level h, then the "direct" level h path has the same maximum delay as a path that goes deeper. A corollary of this, is that all paths $P \in [h, h + k]$ between the two events have the same maximum delay.

Note that the statement of the problem (i.e. that path P exists and $P \in (h, h + k]$), means that the first edge in P is an $r_h \to r_{h+1}$ edge and the last edge in P is an $a_{h+1} \to a_h$ edge. To satisfy this constraint means that $j = i + m \cdot n$ with $m > 0$.
Proof. By induction on k.

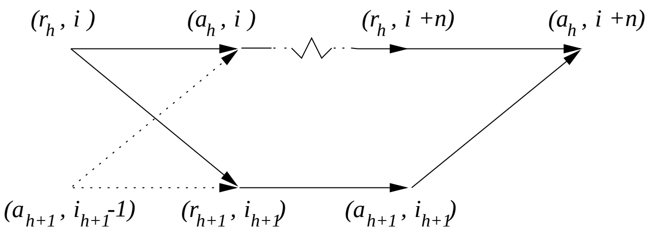

Figure 8: Example of path when $k = 1$ and $m = 1$.

Basis. Let $k = 1$. We prove the base case by a separate induction on m.

Basis. Let $m = 1$. Then $j = i + n$. Refer to figure 8.

We can trace a path directly at level h between $T(r_h, i)$ and $T(a_h, i+n)$. We have $n+1$ $r_h \to a_h$ edges on this path and n $a_h \to r_h$ edges. The delay on each $r_h \to a_h$ is maximized at $\Delta_{h,r}$. Likewise the delay on each $a_h \to r_h$ is at most $\Delta_{h-1,f}$. Then,

$$\begin{aligned} T(a_h, i+n) - T(r_h, i) &= (n+1)\Delta_{h,r} + n \cdot \Delta_{h-1,f} \\ &= (n+1)n^h \Delta_{0,r} + n^h \cdot \Delta_{0,f} \end{aligned}$$

This is the delay on path P'. Now we derive the delay of the path $P \in (h, h+1)$ with $j = i + n$. There is only one such path and it is $r_h \to r_{h+1} \to a_{h+1} \to a_h$. We can calculate the corresponding maximum delay.

$$\begin{aligned} T(a_h, i+n) - T(r_h, i) &= \Delta_{h,f} + \Delta_{h+1,r} + \Delta_{h,r} \\ &= n^h \cdot \Delta_{0,f} + n^{h+1} \cdot \Delta_{0,r} + n^h \cdot \Delta_{0,r} \\ &= (n+1)n^h \Delta_{0,r} + n^h \cdot \Delta_{0,f} \end{aligned}$$

So the delay on P is the same as the delay on P'.

Step $k = 1$ as above. Assume that our property holds for $m = M$. Now we prove for the case when $m + 1$. Refer to figure 9. By the inductive assumption the delay between $T(r_h, i)$ and $T(a_h, i + m \cdot n)$ is the same independent of whether the critical path is $P \in (h, h+1]$, or whether it is $P' \in [h, h]$. Let $T(a_h, i + m \cdot n) - T(r_h, i) = x$.

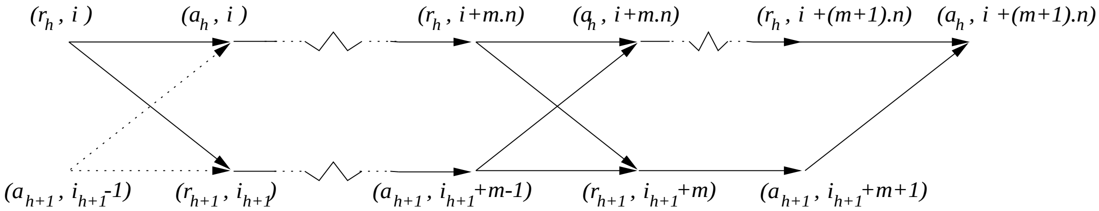

Figure 9: Example of path when $k = 1$ and $m \geq 1$.

Now we derive the delay $T(a_h, i+(m+1)\cdot n) - T(r_h, i)$. First we examine the level h path P'. Between $(a_h, i+m\cdot n)$ and $(a_h, i+(m+1)\cdot n)$ there are n $r_h \to a_h$ and n $a_h \to r_h$ edges.

$$\begin{aligned}
T(a_h, i+(m+1)\cdot n) - T(r_h, i) &= (T(a_h, i+(m+1)\cdot n) - T(a_h, i+m\cdot n)) \\
&\quad +(T(a_h, i+m\cdot n) - T(r_h, i)) \\
&= (T(a_h, i+(m+1)\cdot n) - T(a_h, i+m\cdot n)) + x \\
&= n\cdot\Delta_{h,r} + n\cdot\Delta_{h-1,f} + x \\
&= n^{h+1}\cdot\Delta_{0,r} + n^h\cdot\Delta_{0,f} + x
\end{aligned}$$

Now we examine the path $P \in (h, h+1]$. First we note that the two incoming edges to $(a_h, i+m\cdot n)$ must have maximal delay of $\Delta_{h,r}$, otherwise the inductive hypothesis is contradicted. Thus both predecessors of $T(a_h, i+m\cdot n)$ (i.e. $(r_h, i+m\cdot n)$ and $(a_{h+1}, i_{h+1}+m-1)$) must occur at $T(a_h, i+m\cdot n) - \Delta_{h,r} = T(r_h, i) + x - \Delta_{h,r}$. One of the two predecessors is an acknowledgment event at level $h+1$. Call this event e_0. Now, path P is restricted to events at level $h+1$ except for the endpoints and so must pass through point e_0 (i.e. $(a_{h+1}, i_{h+1}+m-1)$). So we have a path $(r_h, i) \rightsquigarrow e_0 \rightsquigarrow (a_h, i+(m+1)\cdot n)$. The path $e_0 \rightsquigarrow (a_h, i+(m+1)\cdot n)$ consists of one $a_{h+1} \to r_{h+1}$ edge, one $r_{h+1} \to a_{h+1}$ edge, and one $a_{h+1} \to a_h$ edge. These have delays respectively of $\Delta_{h,f}$, $\Delta_{h+1,r}$ and $\Delta_{h,r}$.

$$\begin{aligned}
T(a_h, i+(m+1)\cdot n) - T(r_h, i) &= (T(a_h, i+(m+1)\cdot n) - T(e_0)) + (T(e_0) - T(r_h, i)) \\
&= x - \Delta_{h,r} + (T(e_0) - T(a_h, i+m\cdot n)) \\
&= x - \Delta_{h,r} + (\Delta_{h,f} + \Delta_{h+1,r} + \Delta_{h,r}) \\
&= x + (\Delta_{h,f} + \Delta_{h+1,r}) \\
&= \Delta_{h+1,r} + \Delta_{h,f} + x \\
&= n^{h+1}\cdot\Delta_{0,r} + n^h\cdot\Delta_{0,f} + x
\end{aligned}$$

This is the same as the delay on path P' and we are done. We have now completed the base case for when $k = 1$.

Step. Assume that the property holds for $k = K$. Now prove for $k+1$. We wish to show that if we have a path $P \in (h, h+k+1]$, between (r_h, i) and (a_h, i), and we have a level h path between the two, then the maximum delay on each path is the same.

Note that path P must begin with an $r_h \to r_{h+1}$ edge and finish with an $a_{h+1} \to a_h$ edge. So we have a path $r_h \to r_{h+1} \rightsquigarrow a_{h+1} \to a_h$. The path $r_{h+1} \rightsquigarrow a_{h+1}$ is a path in $[h+1, h+k+1]$, or by setting $f = h+1$, path $r_f \rightsquigarrow a_f$ is a path in $[f, f+k]$.

The path $r_f \rightsquigarrow a_f$ is either a path at level f, or consists of segments S, some of which are paths at level f, and some of which are paths $S \in (f, f+k]$.

Suppose that somewhere on the path $r_f \rightsquigarrow a_f$ there is a segment $(r_f, i_f) \rightsquigarrow (a_f, j_f)$ where $j_f = i_f + m\cdot n$, $m > 0$. Refer to figure 10. By the induction hypothesis, the maximal delay on this segment is identical whether it is a segment at level f, or if $S \in (f, f+k]$. Without loss of generality we assume that segment S is a level f path. Segments $S' = e_0 \rightsquigarrow e_1$ that are not of the above form, cannot possibly be segments $S' \in (h, h+k]$, so must be segments at level f. For each type of segment therefore, there is only one possible maximum delay.

Therefore, we cannot choose a path between r_f and a_f with delay longer than the level f path.

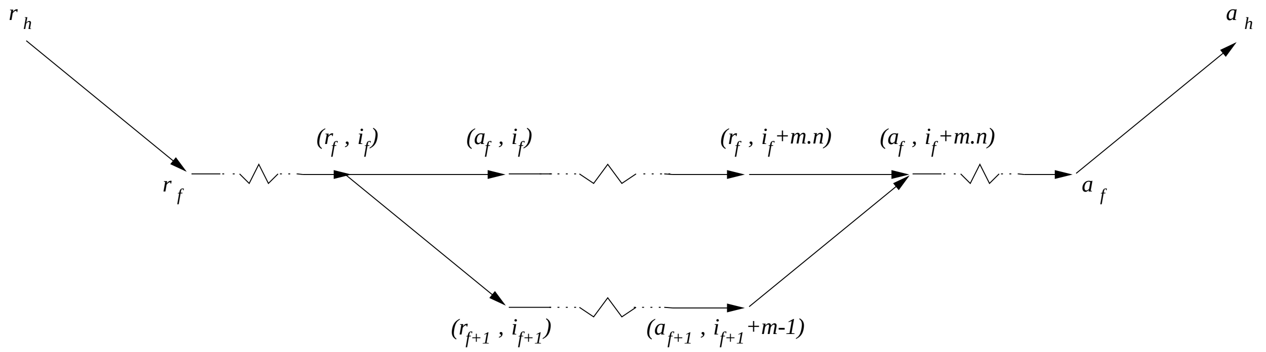

Figure 10: Example of path when $k = 1$ and $m \geq 1$.

So, without loss of generality, we assume that path $r_f \rightsquigarrow a_f$ is a level f path, and therefore path $P \in (h, h+1]$. We now have the base case, which we have already proven and we are done.

Theorem 9 *Let $n > 1$, $\Delta_{i,f} \leq \Delta_{0,f} \cdot n^i$ and $\Delta_{i,r} \leq \Delta_{0,r} \cdot n^i$. Then,*

$$AR \leq \max\{\Delta_{0,r}, \Delta_{0,r} + \Delta_{0,f} - \delta_e\}$$

Proof. Let there be a critical path P, from from event i at level 0 to event j at level 0, with $j \gg i$. If j and i are sufficiently far apart then, without loss of generality we let event i be a request and event j be an acknowledgement such that $j = i + m \cdot n$, for some $m > 0$. Because we are looking a bound for worst-case behaviour we may assume that the bounds on $\Delta_{i,f}$ and $\Delta_{i,r}$ are equality. That is, we let $\Delta_{i,f} = \Delta_{0,f} \cdot n^i$ and $\Delta_{i,r} = \Delta_{0,r} \cdot n^i$.

Path $P \in [0, L]$, but without loss of generality we need only examine path $P \in [0, 1]$ due to theorem 8 above.

Suppose that the critical path P, is a level 0 path. Then $T(a_0, j) - T(r_0, i) \leq (j - i + 1)\Delta_{0,r} + (j - i) \cdot (\text{Environment Delay})$. To determine the amortized response time between (r_0, i) and (a_0, j) we subtract the environment delay and divide by the number of cycles. The number of cycles is $j - i + 1$, so, if the critical path P is a level 0 path the amortized response time $AR_0 \leq \Delta_{0,r}$.

If the critical path P is not a level 0 path then we can break P into segments. There is at least one segment S of the form $S \in (0, 1]$. Let S start at (r_0, i_0) and end at (a_0, j_0). Then $j_0 = i_0 + m' \cdot n$. We can calculate the maximum delay on S.

$$\begin{aligned} T(a_0, i_0 + m' \cdot n) - T(r_0, i_0) &\leq \Delta_{0,f} + m' \cdot \Delta_{1,r} + (m' - 1) \cdot \Delta_{1,f} + \Delta_{0,r} \\ &\leq \Delta_{0,f} + m' \cdot n \cdot \Delta_{0,r} + (m' - 1)n \cdot \Delta_{0,f} + \Delta_{0,r} \end{aligned}$$

We derive the amortized response time over this segment by dividing by subtracting the number of environment delays, and dividing by the number of cycles at level 0.

$$\begin{aligned} AR_m &\leq (\Delta_{0,f} + m' \cdot n \cdot \Delta_{0,r} + (m' - 1)n \cdot \Delta_{0,f} + \Delta_{0,r} - m' \cdot n \cdot \delta_e)/(m' \cdot n + 1) \\ &= (\Delta_{0,f} - \Delta_{0,r} + (m' - 1)n \cdot \Delta_{0,f} + \Delta_{0,r} + \delta_e)/(m' \cdot n + 1) + \Delta_{0,r} - \delta_e \\ &= (\Delta_{0,f} + (m' - 1)n \cdot \Delta_{0,f} + \delta_e)/(m' \cdot n + 1) + \Delta_{0,r} - \delta_e \end{aligned}$$

$$
\begin{aligned}
&= (\Delta_{0,f} + (m' \cdot n + 1) \cdot \Delta_{0,f} - n \cdot \Delta_{0,f} - \Delta_{0,f} + \delta_e)/(m' \cdot n + 1) + \Delta_{0,r} - \delta_e \\
&= (\delta_e - n \cdot \Delta_{0,f})/(m' \cdot n + 1) + \Delta_{0,f} + \Delta_{0,r} - \delta_e
\end{aligned}
$$

First let us consider the case where $\delta_e > n \cdot \Delta_{0,f}$. We maximize over m'.

$$
\begin{aligned}
AR &\leq \max\{(\delta_e - n \cdot \Delta_{0,f})/(m' \cdot n + 1) + \Delta_{0,f} + \Delta_{0,r} - \delta_e \mid 1 \leq m'\} \\
&\quad \{\text{Maximize at } m' = 1\} \\
&= (\delta_e - n \cdot \Delta_{0,f})/(n + 1) + \Delta_{0,f} + \Delta_{0,r} - \delta_e \\
&= (\Delta_{0,f} - n \cdot \delta_e)/(n + 1) + \Delta_{0,r} \\
&< (n \cdot \Delta_{0,f} - n \cdot \delta_e)/(n + 1) + \Delta_{0,r} \\
&< (\delta_e - n \cdot \delta_e)/(n + 1) + \Delta_{0,r} \\
&< (0)/(n + 1) + \Delta_{0,r} \\
&= \Delta_{0,r}
\end{aligned}
$$

Let us now consider the case when $\delta_e \leq n \cdot \Delta_{0,f}$. We maximize over m'.

$$
\begin{aligned}
AR &\leq \max\{(\delta_e - n \cdot \Delta_{0,f})/(m' \cdot n + 1) + \Delta_{0,f} + \Delta_{0,r} - \delta_e \mid 1 \leq m'\} \\
&\leq \max\{(0)/(m' \cdot n + 1) + \Delta_{0,f} + \Delta_{0,r} - \delta_e \mid 1 \leq m'\} \\
&= \Delta_{0,f} + \Delta_{0,r} - \delta_e
\end{aligned}
$$

Combining the above two results, and maximizing over all possible segments, we get as the amortized response time

$$AR \leq \max\{\Delta_{0,r}, \Delta_{0,r} + \Delta_{0,f} - \delta_e\}$$

Proof of (ii). Because all $r_0 \rightarrow a_0$ edges have a minimum delay of $\delta_{0,r}$, we know that $AR \geq \delta_{0,r}$ and we are done.

We sketch a proof that this bound is tight when $\delta_{0,f} \leq \Delta_e$. The minimum amortized response time can occur when all delays except those corresponding to the environment are minimum. The minimum amortized response time occurs when the delay between requests is at its maximum. Therefore, we label all edges of the graph with the minimum delay and except the edges representing the environment with the maximum delay, Δ_e.

If all edges in the graph other than those corresponding to the environment are assigned the minimum delay then we can use the same proof as given for theorem 8 to show that for all $P, P' \in [h, h+k], h > 0$ between the points (r_h, i) and a_h, j, the delay on P is the same as the delay on P'.

We can use the same arguments given in the proof of theorem 9 to show that the our lower bound $AR_{lb} \leq \max\{\delta_{0,r}, \delta_{0,r} + \delta_{0,f} - \Delta_e\}$. We also have that $AR_{lb} \geq \delta_{0,r}$. So when $\delta_{0,f} \leq \Delta_e$, $AR_{lb} = \delta_{0,r}$, and the bound is tight.

Note that we could attempt to show that $AR \geq \max\{\delta_{0,r}, \delta_{0,r} + \delta_{0,f} - \Delta_e\}$. by showing that a path which derives $AR = \delta_{0,r} + \delta_{0,f} - \Delta_e$ exists. The problem here is that each $\delta_{i,r}$ and $\delta_{i,f}$ are bounded from *above* by $\delta_{0,r} \cdot n^i$, and $\delta_{0,f} \cdot n^i$, and we don't know if we have equality. If there is equality at some i, i.e. if $\delta_{i,r} = \delta_{0,r} n^i$ and $\delta_{i,f} = \delta_{0,f} n^i$, then we could construct such a proof.
$\square$.

7 Worst-case Cycle Time

We can also calculate the amortized and worst-case cycle time of a pipeline. Response time was the separation between a request and the corresponding acknowledgment. Cycle time is the time between consecutive requests, i.e. the separation $T(r_0, i+1) - T(r_0, i)$. Obviously the delay of the environment $(T(r_0, i+1) - T(a_0, i))$ is the crucial difference between response time and cycle time.

Theorem 10 *The worst-case cycle time CT for any pipeline (as defined in Section 3 operating under a delay model of Section 4) is bounded from above as*

$$CT \quad \leq \quad \Delta_e + \max(\{\Delta_{k,r} + \Delta_{k-1,f} + \sum_{h=0}^{k-1}(\Delta_{h,r} - \delta_{h,r}) - (\delta_{0,r} + \delta_e)n^k + \delta_{0,r} \mid 1 \leq k \leq L\} \cup \{\Delta_{0,r}\})$$

Proof. This is the same as the bound for RT plus one maximum environment delay Δ_e. □

8 Amortized Cycle Time

Theorem 11 *The amortized cycle time AC of a pipeline with multiplication factor $n = 1$ is bounded as*

$$mc \leq AC \leq \max\{MC, \Delta_{0,r} + \Delta_e\}$$

Proof. We are going to look at a path between (r_0, i_0) and $(r_0, j_0 + 1)$ where $i_0 \leq j_0$. From acknowledgment (a_0, j_0) there is a critical response path to some request (r_0, i_0) with $i_0 \leq j_0$ and on this critical response path there is no other event at level 0. Let the length of this response path be $h = 2k+1$. Notice that if $h = 1$ than $j_0 = i_0$ and we have $T(a_0, j_0) - T(r_0, i_0) \leq \Delta_{0,r}$ and hence a delay between $T(r_0, j_0 + 1) - T(r_0, i_0) \leq \Delta_{0,r} + \Delta_e$.

The rest of the derivation follows in the same way as the derivation for amortized response time (for $n = 1$) except we do not subtract the time taken by the environment. Instead we add one environment delay, Δ_e. Thus we have

$$\begin{aligned} AC &\leq (k \cdot MC + \Delta_{0,r} + \Delta_e)/(k+1) \\ &= (\Delta_{0,r} + \Delta_e - MC)/(k+1) + MC \end{aligned}$$

Let us consider two cases while maximizing over k. If $\Delta_{0,r} + \Delta_e \leq MC$ then $AC \leq MC$. If $\Delta_{0,r} + \Delta_e > MC$ then

$$\begin{aligned} AC &\leq (\Delta_{0,r} + \Delta_e - MC)/(k+1) + MC \\ & \quad \{\text{Maximize when } k = 1\ \} \\ &= (\Delta_{0,r} + \Delta_e - MC)/(2) + MC \\ &= (\Delta_{0,r} + \Delta_e + MC)/(2) \\ &< (\Delta_{0,r} + \Delta_e + \Delta_{0,r} + \Delta_e)/(2) \quad \{\text{Since } \Delta_{0,r} + \Delta_e > MC\} \\ &= \Delta_{0,r} + \Delta_e \end{aligned}$$

and we have proved the upperbound. Next we prove the lowerbound.

The lower bound must clearly be at least $\delta_e + \delta_{0,r}$ which is the path through environment and cell 0. But note that $\delta_e + \delta_{0,r} \leq mc$, and we show below that the lowerbound for the amortized cycle time is equal to mc.

As with the previous proof of lowerbound, we pick a path between P_i between (r_0, i_0) and $(r_0, j_0 + 1)$ where $i_0 \leq j_0$.

We label every edge on path P_i with it's minimum value. Since the separation between (r_0, j_0+1) and (r_0, i_0) is dependent on the longest path between the two, i.e. the critical path between the two, then the separation depends on max(delay_on_path(P_i) | $\forall i$) plus one environment delay. Note that max(delay_on_path(P_i) | $\forall i$) $\leq k \cdot mc + \delta_{0,r} + \delta_e$. The last δ_e term is the minimum delay of the environment which completes the cycle. Let AC_{lb} be the lowerbound for AC.

$$\begin{aligned}
AC_{lb} &\leq \max\{\text{delay_on_path}(P_i))/(k+1) \mid \forall i, k\} \\
&\leq \max\{k \cdot mc + \delta_{0,r} + \delta_e)/(k+1) \mid \forall k\} \\
&= \max\{k \cdot mc + (\delta_{0,r} + \Delta_e) - \Delta_e + \delta_e)/(k+1) \mid \forall k\} \\
&\leq \max\{k \cdot mc + mc - \Delta_e + \delta_e)/(k+1) \mid \forall k\} \\
&= \max\{(k+1) \cdot mc + (\delta_e - \Delta_e))/(k+1) \mid \forall k\} \\
&\leq \max\{(k+1) \cdot mc + 0)/(k+1) \mid \forall k\} \\
&= mc
\end{aligned}$$

This bound can be achieved when $k \to \infty$. When $k \to \infty$ then the length of the paths P_i also tend to infinity. On some path P_i there is an infinite number of cycles with delay equal to mc. This will clearly be the longest path.

Then we have

$$AC \geq mc$$

□

Theorem 12 *Let the maximum delays for each cell be bounded as follows.*

$$\begin{aligned}
\Delta_{i,f} &\leq \Delta_{0,f} \cdot n^i \\
\Delta_{i,r} &\leq \Delta_{0,r} \cdot n^i
\end{aligned}$$

The amortized response time AR for any pipeline (as defined in Section 3 operating under a delay model of Section 4), and with $n > 1$, is bounded from above as follows

$$\text{(i)} \quad AC \leq \Delta_{0,r} + \max\{\Delta_{0,f}, \Delta_e\}$$

Instead, let the maximum delays be unbounded. Let the minimum delays for each cell be bounded as follows.

$$\begin{aligned}
\delta_{i,f} &\leq \delta_{0,f} \cdot n^i \\
\delta_{i,r} &\leq \delta_{0,r} \cdot n^i
\end{aligned}$$

The amortized response time AR for any pipeline (as defined in Section 3 operating under a delay model of Section 4), and with $n > 1$, is bounded from above as follows

$$\text{(ii)} \quad AC \geq \delta_{0,r} + \delta_e$$

Proof of (i). As above, the level 0 path, gives us. $T(r_0, i_0+1) - T(r_0, i_0) \leq \Delta_{0,r} + \Delta_e$.

The rest of the derivation follows in the same way as the derivation for amortized response time (for $n > 1$) except we do not subtract the time taken by the environment. Instead we add one environment delay, Δ_e. Thus we have

$$\begin{array}{rcl} AC & \leq & (\Delta_{0,f} + m' \cdot n \cdot \Delta_{0,r} + (m'-1)n \cdot \Delta_{0,f} + \Delta_{0,r} + \Delta_e)/(m' \cdot n + 1) \\ & = & (\Delta_{0,f} - \Delta_{0,r} + (m'-1)n \cdot \Delta_{0,f} + \Delta_{0,r} + \Delta_e)/(m' \cdot n + 1) + \Delta_{0,r} \\ & = & (\Delta_{0,f} + (m'-1)n \cdot \Delta_{0,f} + \Delta_e)/(m' \cdot n + 1) + \Delta_{0,r} \\ & = & (\Delta_{0,f} + (m' \cdot n + 1) \cdot \Delta_{0,f} - n \cdot \Delta_{0,f} - \Delta_{0,f} + \Delta_e)/(m' \cdot n + 1) + \Delta_{0,r} \\ & = & (\Delta_e - n \cdot \Delta_{0,f})/(m' \cdot n + 1) + \Delta_{0,f} + \Delta_{0,r} \end{array}$$

As with AR we have two cases. First let us consider the case where $\Delta_e \leq n \cdot \Delta_{0,f}$. We maximize over m'.

$$\begin{array}{rcl} AC & \leq & \max\{(\Delta_e - n \cdot \Delta_{0,f})/(m' \cdot n + 1) + \Delta_{0,f} + \Delta_{0,r} \mid 1 \leq m'\} \\ & \leq & \max\{(0)/(m' \cdot n + 1) + \Delta_{0,f} + \Delta_{0,r} \mid 1 \leq m'\} \\ & = & \Delta_{0,f} + \Delta_{0,r} \end{array}$$

Next let us consider $\Delta_e > n \cdot \Delta_{0,f}$. We also maximize over m'.

$$\begin{array}{rcl} AC & \leq & \max\{(\Delta_e - n \cdot \Delta_{0,f})/(m' \cdot n + 1) + \Delta_{0,f} + \Delta_{0,r} \mid 1 \leq m'\} \\ & & \{\text{Maximize at } m' = 1\} \\ & = & (\Delta_e - n \cdot \Delta_{0,f})/(n+1) + \Delta_{0,f} + \Delta_{0,r} \\ & = & (\Delta_e + \Delta_{0,f})/(n+1) + \Delta_{0,r} \\ & < & (\Delta_e + \Delta_e)/(n+1) + \Delta_{0,r} \\ & < & (\Delta_e + n \cdot \Delta_e)/(n+1) + \Delta_{0,r} \\ & = & \Delta_e + \Delta_{0,r} \end{array}$$

By combining all cases we are done. Now we prove the lowerbound.

Proof of (ii). Because all $r_0 \rightarrow a_0$ edges have a minimum delay of $\delta_{0,r}$, and all $a_0 \rightarrow r_0$ edges have a minimum delay of δ_e, we know that $AC \geq \delta_{0,r} + \delta_e$ and we are done.

We sketch a proof that this bound is tight when $\delta_{0,f} \leq \delta_e$. The minimum amortized response time can occur when all delays *including* those corresponding to the environment are minimum.

If all edges in the graph are assigned the minimum delay then we can use the same proof as given for theorem 8 to show that for all $P, P' \in [h, h+k], h > 0$ between the points (r_h, i) and a_h, j, the delay on P is the same as the delay on P'.

We can use the same arguments given in the proof of theorem 9 to show that the our lower bound $AC_{lb} \leq \delta_{0,r} + \max\{\delta_{0,f}, \delta_e\}$. We also have that $AC_{lb} \geq \delta_{0,r} + \delta_e$. So when $\delta_{0,f} \leq \delta_e$, $AC_{lb} = \delta_{0,r} + \delta_e$, and the bound is tight. Note that we could attempt to show that $AC \geq \delta_{0,r} + \max\{\delta_{0,f}, \delta_e\}$ by showing that a path which derives $AC = \delta_{0,r} + \delta_{0,f}$ exists. As with the proof for AR, the problem here is that each $\delta_{i,r}$ and $\delta_{i,f}$ are bounded from *above* by $\delta_{0,r} \cdot n^i$, and $\delta_{0,f} \cdot n^i$, and we don't know if we have equality.

$\square$.