De Morgan Bisemilattices*

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Abstract. We study de Morgan bisemilattices, which are algebras of the form $(S, \sqcup, \wedge, ^{-}, 1, 0)$, where $(S, \sqcup, \wedge)$ is a bisemilattice, 1 and 0 are the unit and zero elements of S, and $^{-}$ is a unary operation, called quasi-complementation, that satisfies the involution law and de Morgan's laws. De Morgan bisemilattices are generalizations of de Morgan algebras, and have applications in multi-valued simulations of digital circuits. We present some basic observations about bisemilattices, and provide a set-theoretic characterization for a subfamily of de Morgan bisemilattices which we call locally distributive de Morgan bilattices.

1 A Quinary Algebra for Logic Circuits

In 1972 Lewis [9, 10] introduced a quinary, that is, five-valued, algebra for simulating the behavior of logic circuits. The five values used were the constant values 0 and 1, the values 0/1 and 1/0 indicating a signal changing from 0 to 1, and from 1 to 0, respectively, and X representing an indeterminate or unknown value. For convenience, we replace 0/1 by U (for *up*) and 1/0 by D (for *down*). In this model, the behaviors of the INVERTER, OR gate and AND gate are described by the operators $^{-}$, $\sqcup$, and $\wedge$, respectively, where 0 is the complement of 1, U is the complement of D, X is its own complement, and $\sqcup$, and $\wedge$ are defined in the tables below.

It is easily verified that the three operations obey the following laws:

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L1	$x \sqcup x = x$	L1$'$	$x \wedge x = x$
L2	$x \sqcup y = y \sqcup x$	L2$'$	$x \wedge y = y \wedge x$
L3	$x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$	L3$'$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$
L4	$x \sqcup 1 = 1$	L4$'$	$x \wedge 0 = 0$
L5	$x \sqcup 0 = x$	L5$'$	$x \wedge 1 = x$
L6	$\overline{\overline{x}} = x$		
L7	$\overline{x \sqcup y} = \overline{x} \wedge \overline{y}$	L7$'$	$\overline{x \wedge y} = \overline{x} \sqcup \overline{y}$

The absorption laws do not hold because $U \wedge (U \sqcup D) = U \wedge X = X \neq U$, and the dual law is violated by the dual argument. Also, the distributive laws do not hold since

$$U \wedge (D \sqcup 1) = U \wedge 1 = U \neq X = X \sqcup U = (U \wedge D) \sqcup (U \wedge 1),$$

and the dual argument applies to the dual law. The operation $^-$ is not a complement operation, since $U \wedge \overline{U} = U \wedge D = X \neq 0$, and, similarly, $U \sqcup \overline{U} \neq 1$.

Table 1: Operation of an OR gate.

$\sqcup$	1	X	D	U	0
1	1	1	1	1	1
X	1	X	X	X	X
D	1	X	D	X	D
U	1	X	X	U	U
0	1	X	D	U	0

Table 2: Operation of an AND gate.

$\wedge$	0	X	U	D	1
0	0	0	0	0	0
X	0	X	X	X	X
U	0	X	U	X	U
D	0	X	X	D	D
1	0	X	U	D	1

In this paper we generalize this example, and study the properties of algebras of the form $(S, \sqcup, \wedge, ^-, 1, 0)$, in which S is a set, $\sqcup$ and $\wedge$ are binary operations on S, $^-$ is a unary operation on S, 1 and 0 are elements of S, and laws L1–L7, and their duals are satisfied.

2 Semilattices

We now briefly review basic properties of semilattices, and establish the notation for the remainder of the paper.

A partially ordered set $(S, \sqsupseteq)$ is 1-*bounded* if it has a *unit*, that is, an element 1 such that $1 \sqsupseteq x$ for all $x \in S$. It is 0-*bounded* if it has a *zero*, that is, an element 0 such that $x \sqsupseteq 0$ for all $x \in S$. A partial order is *bounded* if it is both 1-bounded and 0-bounded.

A partially ordered set $(S, \sqsupseteq)$ is an *upper semilattice* if every pair of elements has a least upper bound $lub_{\sqsupseteq}$. In an upper semilattice, the binary operation $\sqcup$ defined by

$$x \sqcup y = z \text{ iff } z = lub_{\sqsupseteq}\{x, y\}$$

satisfies L1–L3. Equivalently, any set S with a binary operation $\sqcup$ satisfying L1–L3, is an upper semilattice, since we can define the partial order $\sqsupseteq$ by

$$x \sqsupseteq y \text{ iff } x \sqcup y = x,$$

and then $x \sqcup y$ is the least upper bound of $\{x, y\}$.

An upper semilattice that is 1-bounded (respectively, 0-bounded) satisfies L4 (respectively, L5).

An upper semilattice $(S, \sqcup)$ is *complete* [2] if every subset of S has a least upper bound $lub_{\sqsupseteq}$. Note that in a complete upper semilattice $lub_{\sqsupseteq}\emptyset = 0$ and $lub_{\sqsupseteq}S = 1$. Thus, a complete upper semilattice is always bounded. Also, every complete upper semilattice is a complete lattice [2]. For suppose that $(S, \sqcup)$ is a complete upper semilattice, and consider any $T \subseteq S$. Let L be the set of all lower bounds of T. Since S is complete, the element $z = lub_{\sqsupseteq}L$ exists. Now $z = glb_{\sqsupseteq}T$. If we now define

$$x \sqcap y = z \text{ iff } z = glb_{\sqsupseteq}\{x, y\},$$

we have the lattice $(S, \sqcup, \sqcap, 1, 0)$, which we call the lattice *induced* by the complete semilattice $(S, \sqcup, 1, 0)$. Note that every finite bounded semilattice $(S, \sqsupseteq, 1, 0)$ is complete, and hence, a lattice.

Dually, a partially ordered set $(S, \leq)$ is a *lower semilattice* if every pair of elements has a greatest lower bound $glb_{\leq}$. In a lower semilattice, the binary operation $\wedge$ defined by

$$x \wedge y = z \text{ iff } z = glb_{\leq}\{x, y\}$$

satisfies L1′–L3′. Equivalently, any set S with a binary operation $\wedge$ satisfying L1′–L3′, is a lower semilattice, since we can define the partial order $\leq$ by

$$x \leq y \text{ iff } x \wedge y = x,$$

and then $x \wedge y$ is the greatest lower bound of $\{x, y\}$.

A lower semilattice that is 0-bounded (respectively, 1-bounded) satisfies L4′ (respectively, L5′).

A lower semilattice $(S, \leq)$ is *complete* if every subset of S has a greatest lower bound $glb_{\leq}$. Note that in a complete lower semilattice $glb_{\leq}\emptyset = 1$ and $glb_{\leq}S = 0$.

Thus, a complete lower semilattice is always bounded. Every complete lower semilattice is a complete lattice, by an argument dual to the one for upper semilattices. If we define

$$x \vee y = z \text{ iff } z = lub_{\leq}\{x, y\},$$

we have the lattice $(S, \wedge, \vee, 0, 1)$, which we call the lattice *induced* by the complete lower semilattice $(S, \wedge, 0, 1)$.

3 Bisemilattices

Bisemilattices have been studied extensively since 1967; the search for the key-word 'bisemilattice' at `http://www.ams.org/mathscinet/basicsearch` results in 25 papers with that word in the title. In this section we present some basic definitions and observations concerning bisemilattices.

A *bisemilattice*[1] $(S, \sqcup, \wedge)$, consists of a set S with two binary operations $\sqcup$ and $\wedge$ on S, such that $(S, \sqcup)$ and $(S, \wedge)$ are semilattices. In other words, $(S, \sqcup, \wedge)$ satisfies laws L1–L3, L1$'$–L3$'$. Two partial orders are associated with S, as follows. For all $x, y \in S$,

$$x \sqsupseteq y \text{ iff } x \sqcup y = x, \qquad x \leq y \text{ iff } x \wedge y = x.$$

As usual, we use $\sqsubseteq$ and $\geq$ for the converse relations. If two elements a and b of S are not related by $\sqsupseteq$ (respectively $\leq$) we call them $\sqsupseteq$*-incomparable* (respectively $\leq$*-incomparable*). Two elements that are both $\sqsupseteq$-incomparable and $\leq$-incomparable are called *incomparable*.

A bisemilattice $(S, \sqcup, \wedge)$ is *semibounded* if $(S, \sqcup)$ is 1-bounded and $(S, \wedge)$ is 0-bounded. Equivalently, $(S, \sqcup, \wedge)$ is semibounded if it has elements $1_{\sqcup}$ and $0_{\wedge}$ satisfying L4 and L4$'$. Every finite bisemilattice is semibounded. A bisemilattice is *locally bounded* if each of its semilattices is bounded.

In a bisemilattice, the two component semilattices are related only by the fact that they share the same underlying set. If the absorption laws

$$\text{L8} \quad x \sqcup (x \wedge y) = x \qquad \text{L8}' \quad x \wedge (x \sqcup y) = x$$

are satisfied, then the bisemilattice is a lattice, and $x \wedge y$ is the $glb_{\sqsupseteq}\{x, y\}$, that is, $x \wedge y = x \sqcap y$. Similarly, $x \vee y = x \sqcup y$. To verify this, note that $x = x \sqcup (x \wedge y)$ by the absorption law. Thus, $x \sqsupseteq x \wedge y$, and, similarly, $y \sqsupseteq x \wedge y$. Consequently, $x \wedge y$ is a lower bound of x and y with respect to $\sqsupseteq$. If z is any other lower bound of x and y with respect to $\sqsupseteq$, then $x \sqcup z = x$ and $y \sqcup z = y$. Then

[1] In 1967, Płonka [13] introduced the name *quasi-lattice* for an algebra which satisfies L1–L3 and L1$'$–L3$'$. However, he limited his attention to distributive quasi-lattices. In 1971, Padmanabhan [12], used the name *quasilattice* for an algebra satisfying L1–L3 and L1$'$–L3$'$, *and an additional condition.* Padmanabhan then introduced the name *bi-semilattice* for Płonka's quasi-lattice. Following others, we choose the name "bisemilattice", since it is more descriptive than "quasi-lattice".

$z = z \wedge (x \sqcup z) = z \wedge x$, $z = z \wedge y$, and $z = z \wedge y = (z \wedge x) \wedge y = z \wedge (x \wedge y)$. Therefore, $(x \wedge y) \sqcup z = (x \wedge y) \sqcup (z \wedge (x \wedge y)) = x \wedge y$ by absorption. Hence $x \wedge y \sqsupseteq z$, and $x \wedge y$ is indeed the greatest lower bound. Dually, $x \sqcup y$ is the least upper bound with respect to $\leq$.

Also, if the absorption laws are satisfied and the two semilattices are bounded, then $1_\sqcup = 1_\wedge \sqcup 1_\sqcup = 1_\wedge \sqcup (1_\wedge \wedge 1_\sqcup) = 1_\wedge$, and, dually, $0_\sqcup = 0_\wedge$.

A bisemilattice $(S, \sqcup, \wedge)$ is a *bilattice*[2] if $(S, \sqcup)$ and $(S, \wedge)$ are lattices. The lattices are then denoted by $(S, \sqcup, \sqcap)$ and $(S, \wedge, \vee)$.

A bisemilattice $(S, \sqcup, \wedge)$ is *complete* if $(S, \sqcup)$ and $(S, \wedge)$ are complete. As we have stated in the previous section, every complete semilattice is always bounded. Thus, a complete bisemilattice always possesses $0_\sqcup$, $1_\sqcup$, $0_\wedge$, and $1_\wedge$. Moreover, it is always a bilattice. A complete bisemilattice $(S, \sqcup, \wedge)$ has induced lattices $(S, \sqcup, \sqcap, 1_\sqcup, 0_\sqcup)$ and $(S, \wedge, \vee, 0_\wedge, 1_\wedge)$. Every finite locally bounded bisemilattice is complete.

A bilattice is said to be *locally distributive* if both induced lattices are distributive. Note, however, that a locally distributive bilattice need not satisfy

L9 $\quad x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z) \qquad$ L9′ $\quad x \wedge (y \sqcup z) = (x \wedge y) \sqcup (x \wedge z)$

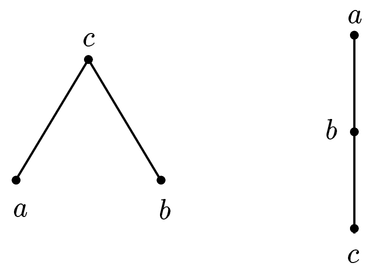
If a bisemilattice satisfies L9 and L9′, it is *distributive* [11].

Example 1 *Figure 1 shows the two partial orders $\sqsupseteq$ and $\leq$ of a three-element bisemilattice $(S, \sqcup, \wedge)$, with $S = \{a, b, c\}$. The semilattice $(S, \sqcup)$ is not a lattice; it is 1-bounded, with $1_\sqcup = c$; it is not complete, since the empty set has no least upper bound. The semilattice $(S, \wedge)$ is a lattice; it is bounded with $0_\wedge = c$, and $1_\wedge = a$. Note that $0_\wedge = 1_\sqcup = c$.* □

Example 2 *Figure 2 shows the partial orders $\sqsupseteq$ and $\leq$ associated with the operations $\sqcup$ and $\wedge$ of Tables 1 and 2. Here, $1_\sqcup = 1_\wedge = 1$ and $0_\sqcup = 0_\wedge = 0$, but the absorption laws do not hold. The algebra is a bilattice. Note that $U \sqcap D = 0$, but $U \wedge D = X$, that is, the two meet operators $\sqcap$ and $\wedge$ are distinct. Since both lattices are distributive, we have a locally distributive bilattice. This bilattice is not distributive, as we have pointed out in Section 1.* □

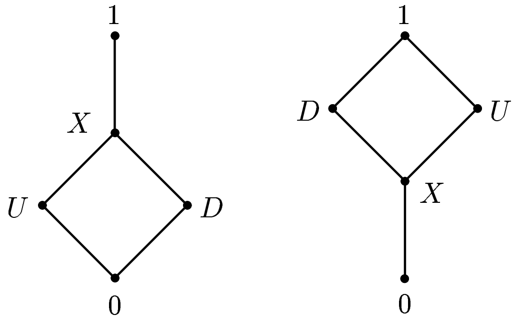
Proposition 1 *Let $(S, \sqcup, \wedge)$ be a bisemilattice and let n be the cardinality of S. Let $N_{bsl}(n)$ and $N_{bl}(n)$ be the numbers of non-isomorphic bisemilattices and bilattices with n elements, respectively. Then $N_{bsl}(1) = N_{bl}(1) = 1$, $N_{bsl}(2) = N_{bl}(2) = 2$, $N_{bsl}(3) = 18$ and $N_{bl}(3) = 8$.*

[2]If two semilattices linked only by the underlying set are called a *bisemilattice*, then consistency of terminology demands that two lattices linked only by the underlying set be called a *bilattice*. The term *2-lattice* has been used for this concept by Romanowska and Trakul [14], but then we would prefer to also use *2-semilattice*. Since the term *bisemilattice* appears to be accepted now, we chose *bilattice*. We point out that the name *bilattice* has been used by Ginsberg [8] and others [14] for different structures, which we discuss later.



(a) (b)

Figure 1: Partial orders for a bisemilattice: (a) $\sqsupseteq$; (b) $\leq$.



(a) (b)

Figure 2: Partial orders for quinary algebra: (a) $\sqsupseteq$; (b) $\leq$.

Proof: Consider the number of distinct bisemilattices.

- If S has one element, we have the **one-element lattice**.
- If S has two elements, we have two cases. If $1_\sqcup \neq 0_\wedge$, then $S = \{1_\sqcup, 0_\wedge\}$. If we simplify the notation to $S = \{1, 0\}$, for the $\sqcup$ operation we must have $1 \sqcup 1 = 1$ and $0 \sqcup 0 = 0$ by L1, $0 \sqcup 1 = 1$ by L4, and $1 \sqcup 0 = 0 \sqcup 1 = 1$ by L2. This is Boolean addition, which is associative. Similarly, $\wedge$ must be Boolean multiplication. Here S is a bilattice; in fact, it is the **two-element lattice**.

  If $1_\sqcup = 0_\wedge$, use a to represent $1_\sqcup = 0_\wedge$, and let b be the second element. We must have $b \sqcup b = b$, $a \sqcup a = a \sqcup b = b \sqcup a = a$, and $b \wedge b = b$, $a \wedge a = a \wedge b = b \wedge a = a$. Thus the two operations are identical, but $a \sqsupseteq b$ and $a \leq b$. Here we have the **two-element bilattice that is not a lattice**.
- If S has three elements, again we have two cases. If $1_\sqcup = 1 \neq 0_\wedge = 0$, let a be the third element. We must have $1 \sqcup 1 = 1$, $a \sqcup a = a$, and $0 \sqcup 0 = 0$ by

L1; $0 \sqcup 1 = a \sqcup 1 = 1$ by L4; $1 \sqcup 0 = 1 \sqcup a = 1$ by L4; and $0 \sqcup a = a \sqcup 0$ by L2. There are three choices for $0 \sqcup a = a \sqcup 0$, namely 1, a, and 0. One verifies that all the choices lead to an associative addition. These choices correspond to the three partial orders (1) $1 \sqsupseteq a, 0$ and a and 0 are $\sqsupseteq$-incomparable, 2) $1 \sqsupseteq a \sqsupseteq 0$, and (3) $1 \sqsupseteq 0 \sqsupseteq a$. Since dual arguments apply to the $\wedge$ operation, we can have (1) $0 \leq a, 1$ and a and 1 are $\leq$-incomparable, (2) $0 \leq a \leq 1$, and (3) $0 \leq 1 \leq a$. Hence, there are altogether nine possibilities. Among these, there are **four bilattices**, one of which is the **three-element lattice**.

In case $1_\sqcup = 0_\wedge$, let the other two elements be a and b. All the entries in the addition and multiplication tables are uniquely determined except the entries for $a \sqcup b = b \sqcup a$ and $a \wedge b = b \wedge a$. As in the case above, there are three choices for $a \sqcup b$ and three choices for $a \wedge b$, again resulting in nine cases. Here, there are also **four bilattices**, but no lattices. □

4 Consistently Bounded Bisemilattices

A *consistently bounded* bisemilattice is a bounded bisemilattice in which $0_\sqcup = 0_\wedge = 0$ and $1_\sqcup = 1_\wedge = 1$. Thus, it is an algebra $(S, \sqcup, \wedge, 1, 0)$, satisfying L1–L5, L1′–L5′. From now on we consider only consistently bounded bisemilattices with at least two elements. Note that all consistently bounded finite bisemilattices are necessarily bilattices.

Proposition 2 *Let $(S, \sqcup, \wedge, 1, 0)$ be a consistently bounded bisemilattice.*

(a) *For all $x, y \in S$, $x \sqcup y = 0$ implies $x = y = 0$, and $x \wedge y = 1$ implies $x = y = 1$.*

(b) *If S is consistently bounded and has at least two elements, then $0 \neq 1$.*

(c) *Let $N_{cb}(n)$ be the number of consistently bounded bisemilattices with n elements. Then $N_{cb}(2) = N_{cb}(3) = 1$, and $N_{cb}(4) = 5$.*

Proof: For (a), suppose x is any element different from 0, and that $x \sqcup y = 0$. Then $x = x \sqcup 0 = x \sqcup (x \sqcup y) = (x \sqcup x) \sqcup y = x \sqcup y = 0$, which is a contradiction. A dual argument proves the second claim. For (b), suppose $0 = 1$, and a is any other element; then $a = a \sqcup 0 = a \sqcup 1 = 1$, which is a contradiction.

Finally, consider (c).

- If $n = 2$, only the **two-element lattice** is consistently bounded.
- For $n = 3$, only the **three-element lattice** is consistently bounded.
- For $n = 4$, let the underlying set be $S = \{0, a, b, 1\}$. The $\sqcup$ and $\wedge$ tables are uniquely defined by the laws, except for the entries $a \sqcup b = b \sqcup a$ and $a \wedge b = b \wedge a$. In view of Proposition 2(a), there are three possibilities for each entry, giving a total of 9 cases to be considered.
  - If $0 \sqsubseteq a, b \sqsubseteq 1$ and $0 \leq a, b \leq 1$, we have the **four-element lattice in which a and b are incomparable**.

- If $0 \sqsubseteq a \sqsubseteq b \sqsubseteq 1$ and $0 \leq a \leq b \leq 1$, we have the **four-element lattice which is totally ordered**. The case $0 \sqsubseteq b \sqsubseteq a \sqsubseteq 1$ and $0 \leq b \leq a \leq 1$ is isomorphic.
- The case $0 \sqsubseteq a \sqsubseteq b \sqsubseteq 1$ and $0 \leq a, b \leq 1$ is isomorphic to $0 \sqsubseteq b \sqsubseteq a \sqsubseteq 1$ and $0 \leq a, b \leq 1$.
- The case $0 \sqsubseteq a, b \sqsubseteq 1$ and $0 \leq a \leq b \leq 1$ is isomorphic to $0 \sqsubseteq a, b \sqsubseteq 1$ and $0 \leq b \leq a \leq 1$. These two cases can be obtained from the two just above by interchanging the $\sqsubseteq$ and $\leq$.
- The case $0 \sqsubseteq a \sqsubseteq b \sqsubseteq 1$ and $0 \leq b \leq a \leq 1$ is isomorphic to $0 \sqsubseteq b \sqsubseteq a \sqsubseteq 1$ and $0 \leq a \leq b \leq 1$. $\square$

5 De Morgan Bisemilattices

A *de Morgan bisemilattice* is an algebra $(S, \sqcup, \wedge, ^-, 1, 0)$, where $(S, \sqcup, \wedge, 1, 0)$ is a consistently bounded bisemilattice and $^-$ is a unary operation on S, called *quasi-complement*, which satisfies the involution law L6 and de Morgan's laws L7, L7$'$. Thus, a de Morgan bisemilattice satisfies L1–L7, L1$'$–L7$'$. A de Morgan bisemilattice is a *de Morgan algebra* if it is a distributive lattice, that is, if it satisfies L8, L8$'$, L9, and L9$'$.

An element a of a de Morgan bisemilattice is *self-complementary* if $\overline{a} = a$.

Proposition 3 *The following holds in a bisemilattice $B = (S, \sqcup, \wedge, ^-, 1, 0)$ with a unary operation $^-$:*

(a) *If B is de Morgan, then $a \leq b$ iff $\overline{a} \sqsupseteq \overline{b}$, and $\overline{0} = 1$.*

(b) *If B satisfies L6 and for all $a, b \in S$, $a \leq b$ iff $\overline{a} \sqsupseteq \overline{b}$, then B is de Morgan.*

(c) *If B is de Morgan, then $a \neq b$, $a \sqsubseteq b$, and $a \leq b$ implies that either $a \neq \overline{a}$ or $b \neq \overline{b}$.*

(d) *If B is de Morgan, then $a \neq b$, $a \sqsupseteq b$, and $a \leq b$ implies $a \neq \overline{b}$.*

(e) *Let $N_{dM}(n)$ be the number of de Morgan bisemilattices with n elements. Then $N_{dM}(2) = N_{dM}(3) = 1$, and $N_{dM}(4) = 3$.*

Proof:

For (a), suppose $a \leq b$; then $a \wedge b = a$, and $\overline{a} = \overline{a \wedge b} = \overline{a} \sqcup \overline{b}$, that is $\overline{a} \sqsupseteq \overline{b}$. The second claim now follows.

For (b), if $x \sqcup y = z$, then z is the $lub_{\sqsubseteq}$ of x and y. By the condition $a \leq b$ iff $\overline{a} \sqsupseteq \overline{b}$, $\overline{z}$ is a lower bound in the partial order $\leq$ of $\overline{x}$ and $\overline{y}$. If it is not the greatest lower bound, this would contradict that z is the least upper bound, again by the condition $a \leq b$ iff $\overline{a} \sqsupseteq \overline{b}$.

For (c), by Part (a) we have that $a \sqsubseteq b$ implies $a = \overline{a} \geq \overline{b} = b$, a contradiction.

For (d), suppose $a \neq b$, $a \sqsupseteq b$, $a \leq b$, and $a = \overline{b}$. Then, by Proposition 3(b), $\overline{a} \sqsupseteq \overline{b}$, and so $b \sqsupseteq a$, which is a contradiction.

Finally, consider Part (e).

- If $n = 2$, by Proposition 3(a), we have the **two-element Boolean algebra**.
- If $n = 3$, let $S = \{0, a, 1\}$. Since there is only one consistently bounded bisemilattice, it only remains to determine the quasi-complements. Since $\overline{0} = 1$, we must have $\overline{a} = a$. Here we have the **three-element ternary algebra** [4, 5].
- For $n = 4$, let $S = \{0, a, b, 1\}$. Suppose a and b are incomparable in both partial orders. By Proposition 2(a) we must have $a \sqcup b = 1$ and $a \wedge b = 0$. If $a = \overline{b}$, we have the **four-element Boolean algebra**.

  If a and b are incomparable, and a and b are self-complementary, we have the **four-element subdirectly irreducible de Morgan algebra** [2].

  Suppose $a \leq b$ and $a \sqsubseteq b$. By Proposition 3(c), either a or b is not self-complementary. Suppose a is that element; then b is the only possible quasi-complement of a. The same is true if b is not self-complementary. This de Morgan bisemilattice is isomorphic to the **de Morgan algebra**

  $$(\{0, .25, .75, 1\}, \max, \min, \bar{\ }, 1, 0),$$

  where max and min are the maximum and minimum operations, and $\overline{x} = 1 - x$.

  The case where $a \leq b$, but a and b are $\sqsupseteq$-incomparable cannot occur. For suppose $a = \overline{a}$ and $b = \overline{b}$. Since $a \leq b$, we have $\overline{a} \sqsupseteq \overline{b}$, by Proposition 3(b). But then $a \sqsupseteq b$, which is a contradiction. The only other possibility is $a = \overline{b}$. Since $a \leq b$, we have $\overline{a} \sqsupseteq \overline{b}$, that is, $b \sqsupseteq a$, which is again a contradiction.

  The case $a \leq b$ and $a \sqsupseteq b$ cannot occur. By Proposition 3(d), $a \neq \overline{b}$. Hence $a = \overline{a}$ and $b = \overline{b}$ are the only possibilities. But then $b = \overline{a} = \overline{a \wedge b} = \overline{a} \sqcup \overline{b} = a \sqcup b = a$, which is a contradiction. $\square$

The proposition above shows that, for $n \leq 4$, every de Morgan bisemilattice is a de Morgan algebra. The example of Section 1 shows that there is a five-element de Morgan bisemilattice which is not a de Morgan algebra. De Morgan bisemilattices with 2, 3, and 4 elements are shown in Figure 3. Self-complementary elements are shown in boldface type.

6 Locally Distributive De Morgan Bilattices

A *de Morgan bilattice* is a de Morgan bisemilattice $(S, \sqcup, \wedge, \bar{\ }, 1, 0)$ which is a bilattice[3]

[3]Note that a de Morgan bilattice is quite different from Ginsberg's bilattice [8]; the latter (in our notation) is an algebra $(S, \sqcap, \sqcup, \wedge, \vee, \bar{\ })$, where $(S, \sqcap, \sqcup)$ and $(S, \wedge, \vee)$ are complete lattices, and $\bar{\ }$ is a unary operation satisfying $\overline{\overline{x}} = x$, $\overline{x \sqcup y} = \overline{x} \sqcap \overline{y}$, $\overline{x \sqcap y} = \overline{x} \sqcup \overline{y}$, $\overline{x \vee y} = \overline{x} \vee \overline{y}$, and $\overline{x \wedge y} = \overline{x} \wedge \overline{y}$. A somewhat more restricted version of Ginsberg's bilattice was used by Fitting [7].

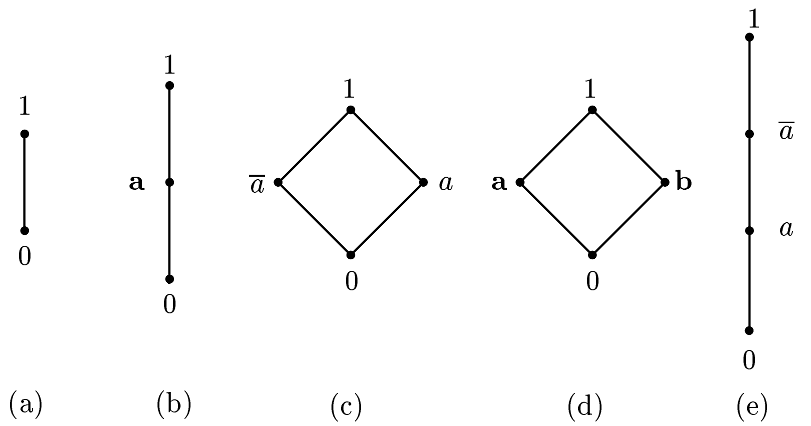


Figure 3: De Morgan bisemilattices with 2, 3, and 4 elements.

A de Morgan bilattice is *locally distributive* if its underlying bilattice has that property. We now derive a characterization of such de Morgan bilattices; this characterization is somewhat similar to the characterizations of ternary algebras in [4, 6], de Morgan algebras in [3], and distributive quasi-lattices (distributive bisemilattices) in [1].

Suppose we are given a locally distributive de Morgan bilattice $(S, \sqcup, \wedge, ^-, 1, 0)$. Since the lattice $(S, \sqcup, \sqcap, 1, 0)$ is distributive, by Birkhoff's theorem [2], it can be embedded in a ring of sets. Thus there exists a set I such that $(S, \sqcup, \sqcap, 1, 0)$ is isomorphic to $(X, \cap, \cup, \emptyset, I)$, under the correspondence $s \mapsto \sigma(s)$, where X is a set of subsets of I, and $\sigma(s)$ denotes the set assigned to s by the embedding. Here $\sqcup$ corresponds to intersection, $\sqcap$ to union, 1 to $\emptyset$, and 0 to I. In general, X is not closed under set complementation.

We can also assign to each element $s \in S$ the set $\overline{\sigma}(s) = \sigma(\overline{s})$. In a de Morgan bisemilattice, the assignment $x \mapsto \overline{x}$ satisfies $x \leq y$ iff $\overline{x} \sqsupseteq \overline{y}$. Now $x \wedge y = z$ iff $\overline{x \wedge y} = \overline{z}$ iff $\overline{x} \sqcup \overline{y} = \overline{z}$ iff $\sigma(\overline{x}) \cap \sigma(\overline{y}) = \sigma(\overline{z})$ iff $\overline{\sigma}(x) \cap \overline{\sigma}(y) = \overline{\sigma}(z)$. Thus, $(S, \wedge, \vee, 0, 1)$ is isomorphic to $(X, \cap, \cup, \emptyset, I)$ under the correspondence $s \mapsto \overline{\sigma}(s)$. Here $\wedge$ corresponds to intersection, $\vee$ to union, 0 to $\emptyset$, and 1 to I.

If we use the double embedding, where each element $s \in S$ corresponds to an ordered pair $(\sigma(s), \overline{\sigma}(s))$ of subsets of I, we have the following properties. For all $x, y, z \in S$,

$$
\begin{aligned}
&x \sqcup y = z \text{ iff } \sigma(x) \cap \sigma(y) = \sigma(z),\\
&x \wedge y = z \text{ iff } \overline{\sigma}(x) \cap \overline{\sigma}(y) = \overline{\sigma}(z),\\
&\overline{x} = y \text{ iff } (\sigma(x), \overline{\sigma}(x)) = (\overline{\sigma}(y), \sigma(y)).
\end{aligned}
$$

Example 3 *We return to the five-element bisemilattice of Section 1. As seen from Figure 2, this is a bilattice. Note that de Morgan's laws are not satisfied by either lattice. For example, in the lattice corresponding to* $\sqsubseteq$, *we have* $\overline{U \sqcup D} = \overline{X} = X$, *whereas* $\overline{U} \sqcap \overline{D} = D \sqcap U = 0$, *and* $\overline{U \sqcap D} = \overline{0} = 1$, *whereas* $\overline{U} \sqcup \overline{D} = D \sqcup U = X$. *Hence this is not a bilattice in Ginsberg's sense.*

To illustrate our characterization, consider the assignment $0 \mapsto (\{1,2,3\}, \emptyset)$, $X \mapsto (\{1\},\{1\})$, $U \mapsto (\{1,2\},\{1,3\})$, $D \mapsto (\{1,3\},\{1,2\})$, $1 \mapsto (\emptyset, \{1,2,3\})$, *as shown in Figure 4.*

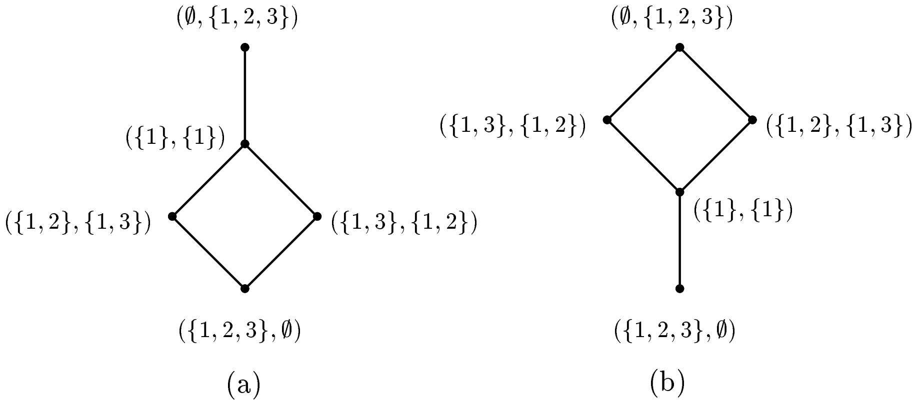


Figure 4: Set embedding for quinary algebra: (a) $\sqsupseteq$; (b) $\leq$.

We can use this representation to simulate circuit behavior as follows. Suppose x *is the input of an inverter and* y *is its output. If* x *is represented by* $(\sigma(x), \overline{\sigma}(x))$, *then the output of the inverter is calculated by simply changing the order of the two components, that is* y *corresponds to* $(\overline{\sigma}(x), \sigma(x))$. *Next, suppose the inputs to an* OR *gate are* $(\sigma(x), \overline{\sigma}(x))$ *and* $(\sigma(y), \overline{\sigma}(y))$. *The first component of the output is then* $C = \sigma(x) \cap \sigma(y)$. *The second component can be calculated by table look-up. First find the element of* S *corresponding to* C. *There is exactly one such element, since the mapping* σ *is an isomorphism, and this element is* $\sigma^{-1}(C)$. *We can then apply the second mapping* $\overline{\sigma}$. *In summary,*

$$(A, B) \sqcup (A', B') = (A \cap A', \overline{\sigma}(\sigma^{-1}(A \cap A'))),$$

$$(A, B) \wedge (A', B') = (\sigma(\overline{\sigma}^{\,-1}(B \cap B')), B \cap B'),$$

and

$$\overline{(A, B)} = (B, A).$$

The dual set assignment, where both $\sqcup$ *and* $\wedge$ *are represented by union, is* $0 \mapsto (\emptyset, \{1,2,3\})$, $X \mapsto (\{1,2\},\{1,2\})$, $U \mapsto (\{1\},\{2\})$, $D \mapsto (\{2\},\{1\})$, $1 \mapsto (\{1,2,3\}, \emptyset)$.

□

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